Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

29 October 2024

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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A004

5 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by 10% per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n , the predicted number of trout at the start of year n, and T_{n+1}, the predicted number of trout at the start of year $n+1$, is given by

$$T_{n+1} = 1.1T_n - 500 \text{ and } T_1 = 6000.$$

For example, the predicted number of trout at the start of the second year is given by

$$T_2 = 1.1T_1 - 500.$$

(a) Use this formula to verify that $T_2 = 6100$. [2]

(b) (i) Verify that $T_3 = 6210$. [1]

(ii) Find T_4. [2]

It is also known that $T_n = 6000(1.1)^{n-1} - \dfrac{500\left((1.1)^{n-1} - 1\right)}{1.1 - 1}$.

(c) (i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

(ii) Hence, or otherwise, find T_6. Give your answer to the nearest whole number. [2]

(This question continues on the following page)

(Question 1 continued)

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n, the predicted number of trout at the start of year n, and D_{n+1}, the predicted number of trout at the start of year $n+1$, is now given by

$$D_{n+1} = 1.1D_n - 750 \text{ and } D_1 = 6000.$$

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

(d) (i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

(ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

(e) Determine the first year during which there will be no trout remaining in the lake. [4]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the nth year where

$$C_n = 6000(1.1)^{n-1} - 10d\left((1.1)^{n-1} - 1\right).$$

(f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$$u_{n+1} = ru_n - d, \text{ where } d, r \in \mathbb{R}^+ \text{ and } r \neq 1.$$

(g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \frac{d\left(r^{n-1} - 1\right)}{r-1}$, for $n \in \mathbb{Z}^+$. [7]

2. [Maximum mark: 28]

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a, b, a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	Sequence of coefficients
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a "reciprocal pair", since one root is the reciprocal of the other.

(a) (i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

(ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p(x) = x^n p\left(\frac{1}{x}\right)$.

(d) Use $p(x) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

(e) Show that f is a palindromic polynomial. [4]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$$f(x) = (x^2 + ux + 1)(x^2 + vx + 1)\text{, where } u, v \in \mathbb{Z} \text{ and } u < v.$$

(f) By forming and solving an appropriate system of equations in u and v, determine the value of u and the value of v. [6]

(g) Hence, find all the exact complex and purely real roots of $x^4 + 2x^3 - x^2 + 2x + 1 = 0$. [3]

Consider the palindromic polynomial equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_2x^2 + a_1x + 1 = 0\text{, where } n \text{ is odd.}$$

(h) Show that -1 is always a root of this equation. [4]